Topology Vol. 13, pp. 313–326. Pergamon Press 1974. Printed in Great Britain

THE SECOND COHOMOLOGY GROUP OF G WITH $\mathbf{Z}_2 G$ COEFFICIENTS

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(*Received* 9 *April* 1974)

§0. INTRODUCTION

THE FOLLOWING result was proven by Hopf [8] and Freudenthal [6]. Here $\mathbf{Z}_2$ denotes the ring of integers modulo 2.

THEOREM. *If G is a finitely presented group, then the dimension of the $\mathbf{Z}_2$ vector space $H^1(G,\mathbf{Z}_2 G)$ is* 0, 1, *or* ∞.

Our main result which generalizes this theorem is

THEOREM 1.5. *If G is a finitely presented group which contains at least one element of infinite order, then any sub-G-module of $H^2(G,\mathbf{Z}_2 G)$ has dimension* 0, 1 *or* ∞ *when considered as a $\mathbf{Z}_2$ vector space.*

The following is a brief outline of this paper's contents. §1 is devoted to proving Theorem 1.5. In §2 and §4, we study the algebraic structure of groups G such that $H_2(G,\mathbf{Z}_2 G)$ has $\mathbf{Z}_2$-dimension 1 and ∞ respectively. Unfortunately we obtain only partial results in this study. §3 contains an application of a spectral sequence technique, developed in §2, to the study of crystallographic groups.

Professor J-P. Serre asked the following question after reading an earlier version of this paper.

Question. Let G be a finitely presented group, K a field, and n a non-negative integer. Do all sub-G-modules of $H^n(G,KG)$ have dimension 0, 1, or ∞ when considered as K vector spaces?

It is known that this question has an affirmative answer when $n = 1$. In the Appendix we show how to generalize the arguments of §1 to give an affirmative answer when $n = 2$, G contains at least one element of ∞ order, and K has non-zero characteristic.†

This article is the result of our unsuccessful attempt to study the conjecture of Eilenberg and Ganea [4] which states that if G has cohomological dimension 2, then its geometric dimension is also 2.

I wish to take this opportunity to thank R. Wells and S. Armentrout for many helpful conversations during the preparation of this paper, and also to thank J-P. Serre for his pertinent comments on an earlier version of it which have been very useful in improving our exposition and emphasis.

† See the author's forthcoming paper "Poincaré duality and groups of type (FP)" for some additional results on this question.

§1. THE MAIN RESULT

If X is a locally finite simplicial complex, then the cohomology with $\mathbf{Z}_2$ coefficients of the ends of X, denoted $H_e^*(X,\mathbf{Z}_2)$, is defined to be the cohomology of the singular $\mathbf{Z}_2$-cochains modulo the $\mathbf{Z}_2$-cochains with compact support. (See page 5 of [15].) An alternate description of $H_e^*(X,\mathbf{Z}_2)$ is the cohomology of the simplicial $\mathbf{Z}_2$-cochains modulo the finite simplical $\mathbf{Z}_2$-cochains.

Notation 1.1. The universal cover of a topological space X is denoted by $\tilde{X}$.

PROPOSITION 1.2. *If G is a finitely presented group and X is any connected, finite, simplicial complex such that $\pi_1 X$ is G, then $H^2(G,\mathbf{Z}_2 G)$ and $H_e^1(\tilde{X},\mathbf{Z}_2)$ are isomorphic G-modules.*

Proof. The short exact sequence of G-bimodules $0 \to \mathbf{Z}_2 \to \overline{\mathbf{Z}_2 G} \to \mathscr{E}G \to 0$ yields the following long exact sequence of G-modules in cohomology:

$$\cdots \to H^i(G,\mathbf{Z}_2 G) \to H^i(G,\overline{\mathbf{Z}_2 G}) \to H^i(G,\mathscr{E}G) \to H^{i+1}(G,\mathbf{Z}_2 G) \to \cdots.$$

(See page 319 of [16] for the definitions of $\overline{\mathbf{Z}_2 G}$ and $\mathscr{E}G$.) Since $H^i(G,\overline{\mathbf{Z}_2 G})$ vanishes for $i > 0$, $H^2(G,\mathbf{Z}_2 G)$ and $H^1(G,\mathscr{E}G)$ are isomorphic G-modules. Let C_i denote the ith simplicial chain group of $\tilde{X}$. Since the complex

$$\cdots \to C_i \to C_{i-1} \to \cdots \to C_0 \to \mathbf{Z} \to 0$$

is a free G-chain complex which is exact at C_1, C_0 and $\mathbf{Z}$, we need only change this complex in dimensions greater than 2 in order to obtain a free G-resolution of the trivial G-module $\mathbf{Z}$. Therefore, $H^1(G,\mathscr{E}G)$ and $H^1(X;\mathscr{E}G)$ are isomorphic G-modules.

Notation 1.3. We donote cohomology with local coefficients by $H^*(;)$.

Since C_i is a finitely generated, free G-module, we obtain the following exact sequence of G-modules:

$$0 \to \mathrm{Hom}_{\mathbf{Z}G}(C_i,\mathbf{Z}_2 G) \to \mathrm{Hom}_{\mathbf{Z}G}(C_i,\overline{\mathbf{Z}_2 G}) \to \mathrm{Hom}_{\mathbf{Z}G}(C_i,\mathscr{E}G) \to 0.$$

By [16], $\mathrm{Hom}_{\mathbf{Z}G}(C_*,\overline{\mathbf{Z}_2 G})$ can be identified with the simplicial $\mathbf{Z}_2$-cochains on $\tilde{X}$, and $\mathrm{Hom}_{\mathbf{Z}G}(C_*,\mathbf{Z}_2 G)$ can be identified with the finite simplicial $\mathbf{Z}_2$-cochains on $\tilde{X}$. Both identifications are G-module isomorphisms. Hence, $H^1(X;\mathscr{E}G)$ and $H_e^1(\tilde{X},\mathbf{Z}_2)$ are isomorphic G-modules.

Notation 1.4. If M is a $\mathbf{Z}_2 G$ module, then the dimension of M refers to its dimension as a $\mathbf{Z}_2$ vector space.

We now state our main result.

THEOREM 1.5. *If G is a finitely presented group which contains at least one element of infinite order, then any sub-G-module of $H^2(G,\mathbf{Z}_2 G)$ has dimension* 0, 1, *or* ∞.

Remark 1.6. If G is a finite group, then $\mathbf{Z}_2 G$ and $\overline{\mathbf{Z}_2 G}$ are identical; hence $H^i(G,\mathbf{Z}_2 G)$ vanishes for all $i > 0$. In particular, Theorem 1.5 is also true when G is a finite group.

COROLLARY 1.7. *If G is a finitely presented group which contains at least one element of infinite order, then the dimension of $H^2(G,\mathbf{Z}_2 G)$ is* 0, 1, *or* ∞.

Notation 1.8. We denote cohomology with compact supports by $H_c^*(\ ,\ )$.

We will deduce Theorem 1.5 from the following proposition.

PROPOSITION 1.9. *If G is a finitely presented group which contains at least one element of infinite order, and M is a closed, connected, smooth manifold of dimension greater than 5 and whose fundamental group is isomorphic to G, then any sub-G-module of $H_c^2(\tilde{M},\mathbf{Z}_2)$ has dimension* 0, 1, *or* ∞.

Proof of Theorem 1.5. It is well known that given any finitely presented group G and integer $n > 5$, there exists a closed, connected, smooth manifold M whose dimension is n, and whose fundamental group is G. Choose one such M. Since M is a finite polyhedron, we have the following long exact sequence of G-modules:

$$\cdots \to H^1(\tilde{M},\mathbf{Z}_2) \to H_e^1(\tilde{M},\mathbf{Z}_2) \to H_c^2(\tilde{M},\mathbf{Z}_2) \to \cdots .$$

Since $H^1(\tilde{M},\mathbf{Z}_2)$ vanishes, we see that $H_e^1(\tilde{M},\mathbf{Z}_2)$ is isomorphic to a sub-G-module of $H_c^2(M,\mathbf{Z}_2)$. Hence ,Theorem 1.5 is a consequence of Proposition 1.9 together with Proposition 1.2.

The proof of Proposition 1.9 depends on the following four lemmas.

Notation 1.10. If g is an element of $\pi_1 X$, then $\tilde{X}/g$ denotes the covering space of X corresponding to the subgroup of $\pi_1 X$ generated by g, and $p:\tilde{X} \to \tilde{X}/g$ denotes the covering projection.

LEMMA 1.11. *Let X be a connected, finite, simplicial complex, g an element of infinite order in $\pi_1 X$, U an open subset of $\tilde{X}/g$ with $\bar{U} - U$ compact, but $\bar{U}$ not compact, and C any compact subset of $\tilde{X}$; then there exists an element h in $\pi_1 X$ such that hC is a subset of $p^{-1}U$.*

Proof. The simplicial structure on X induces simplicial structures on $\tilde{X}$ and $\tilde{X}/g$. Since Lemma 1.11 is obviously true when U is equal to $\tilde{X}/g$, we can assume that U is a proper subset of $\tilde{X}/g$. Also, since C is compact, it is contained in a finite subcomplex of $\tilde{X}$; hence, we may assume that C is a finite subcomplex of $\tilde{X}$.

For each vertex x of $\tilde{X}/g$, define $d(x)$ to be the minimum number of 1-simplexes in a connected simplicial path which starts at x and ends at a vertex of a simplex meeting $\tilde{X}/g - U$. Since $\bar{U} - U$ is compact, while $\bar{U}$ is not compact, and $\tilde{X}/g$ is locally finite, we see that there exist vertices x in U such that $d(x)$ is arbitrarily large.

Let d_1, respectively d_2, be an integer such that any two vertices of C, respectively X, can be joined by a connected simplicial path consisting of d_1, respectively d_2, or fewer 1-simplexes. Let x be a vertex in U such that $d(x) > d_1 + d_2$, x^* a lifting of x to $p^{-1}U$, and y a vertex in C. Denote the covering projection from $\tilde{X}$ to X by r, and let α be a connected simplicial path, consisting of d_2 or fewer 1-simplexes, joining $r(x^*)$ to $r(y)$. Then, α can be lifted to a connected simplicial path α^* in $\tilde{X}$, consisting of d_2 or fewer 1-simplexes, joining x^* to some vertex v where $r(v)$ is equal to $r(y)$; therefore, hC is a subset of $p^{-1}U$ where h is the element in $\pi_1 X$ such that $hy = v$.

LEMMA 1.12. *If K is a connected, locally finite, simplicial complex, and x is an element in $H_c^1(K,\mathbf{Z}_2)$ which maps to 0 in $H^1(K,\mathbf{Z}_2)$, then there exists a proper map $f:K \to R$ such that $f^*(u) = x$, where R denotes the real line, and u generates $H_c^1(\mathbf{R},\mathbf{Z}_2)$.*

Proof. Represent x by a finite, simplicial 1-cocycle P, let E be an infinite 0-cochain such that $\delta E = P$, and note that $\delta(1 - E) = P$ and $1 - E$ is infinite. (We use here the conventions about $\mathbf{Z}_2$ cochains found in §2 of [16].)

We will regard $\mathbf{R}$ as a simplicial complex whose vertices are the integers in $\mathbf{R}$, and whose 1-simplexes are the closed intervals $[n,n + 1]$ where n is an integer. We define a simplicial map $f: K \to \mathbf{R}$ as follows: The vertices of P that are in E are sent to 0, and the ones in $1 - E$ are sent to 1. If x is a vertex of E, define $f(x)$ to equal minus the minimum of the number of 1-simplexes in a connected simplicial path joining x to a vertex of P; and, if x is a vertex of $1 - E$, define $f(x)$ to equal 1 plus the minimum of the number of 1-simplexes in a connected simplicial path joining x to a vertex of P. This vertex correspondence clearly defines a simplicial map $f: K \to \mathbf{R}$, the finiteness of P together with the local finiteness of K imply that f is proper, and it is easy to verify that $f^*(u) = x$.

LEMMA 1.13. *If M is a connected, smooth, orientable manifold of dimension m greater than 5 such that $\pi_1 M$ is infinite cyclic, and the map from $H_c^1(M,\mathbf{Z}_2)$ to $H^1(M,\mathbf{Z}_2)$ is not monic (i.e. M has more than 1 end), then there exists a closed, connected, codimension one submanifold S of M which divides M into two connected components U_1 and U_2, such that neither $\overline{U}_1$ or $\overline{U}_2$ is compact, and the inclusion map of S into M induces an isomorphism on fundamental groups.*

Proof. By Lemma 1.12, there exists a proper map $f: M \to \mathbf{R}$ such that $f^*(u) \neq 0$, where u generates $H_c^1(\mathbf{R},\mathbf{Z}_2)$. Let v be a regular value of f, and N denote $f^{-1}(v)$, then N is a closed, codimension one submanifold of M with trivial normal bundle.

Let $[N]$ denote the fundamental class of N, and $[N]'$ its image in $H_{m-1}(M,\mathbf{Z}_2)$, then $[N]'$ is the Poincaré dual of $f^*(u)$. Let $N_1, N_2, \ldots, N_n$ denote the connected components of N, and $x_1, x_2, \ldots, x_n$ denote the cohomology classes in $H_c^1(M,\mathbf{Z}_2)$ which are the Poincaré duals of $[N_1]', [N_2]', \ldots, [N_n]'$ respectively, then $f^*(u) = x_1 + x_2 + \ldots + x_n$.

We wish to construct a closed, connected, codimension one submanifold T of M with trivial normal bundle such that if y denotes the Poincaré dual of $[T]'$, then y is non-zero, but the image of y in $H^1(M,\mathbf{Z}_2)$ is zero. If we cannot choose one of the N_i to be T, then each non-zero x_i maps to the unique non-zero element of $H^1(M,\mathbf{Z}_2)$. As a result, there exists $i \neq j$ such that $x_i \neq x_j$, but $x_i - x_j$ maps to zero in $H^1(M,\mathbf{Z}_2)$. Let T be the connected sum of N_i and N_j, which can be formed inside of M; then T satisfies the above conditions, and hence T divides M into two unbounded components.

We can modify T by exchanging a finite number of 1 and 2 handles between its sides to obtain a new, codimension one submanifold S which satisfies the conditions of Lemma 1.13. (See page 325 of [5] for more details on this exchanging of handles.)

LEMMA 1.14. *If M is a closed, connected, smooth manifold whose fundamental group G contains an element of infinite order, and $H_c^2(\tilde{M},\mathbf{Z}_2)$ contains a sub-G-module A whose dimension n is finite but greater than 1, then G contains an element g of infinite order such that $\tilde{M}/g$ is an orientable manifold with more than 1 end.*

Proof. Let h be an element of infinite order in G, then the action of h on A determines an element in the finite group $GL_n(\mathbf{Z}_2)$. Hence, there exists an integer m such that h^m acts trivially on A. Let g be h^{2m}, then $\tilde{M}/g$ is an orientable manifold, and $\pi_1\tilde{M}/g$ is an infinite cyclic group. Denote this group by J.

Let Y be the cartesian product of $\tilde{M}$ and $\mathbf{R}$ balanced over J. Explicitly, Y is the quotient of $\tilde{M} \times \mathbf{R}$ via the identifications $(g^i x, r) = (x, r - i)$ where $x \in \tilde{M}$, $r \in \mathbf{R}$, and i is an integer. Then Y fibers over $\tilde{M}/g$ with fiber $\mathbf{R}$, and Y fibers over the circle with fiber $\tilde{M}$.

Since the first fibration is a principal bundle, Y is homeomorphic to $\tilde{M}/g \times \mathbf{R}$, and thus $H_c^1(\tilde{M}/g, \mathbf{Z}_2)$ is isomorphic to $H_c^2(Y, \mathbf{Z}_2)$.

On the other hand, if we apply the Wang sequence, for cohomology with compact supports, to the second fibration, then we obtain the following exact sequence:

$$\cdots \to H_c^2(Y, \mathbf{Z}_2) \to H_c^2(\tilde{M}, \mathbf{Z}_2) \xrightarrow{1-g^*} H_c^2(\tilde{M}, \mathbf{Z}_2) \to \cdots.$$

Since A is contained in the kernel of $1 - g^*$, $H_c^2(Y, \mathbf{Z}_2)$ has dimension bigger than 1. But the dimension of $H_c^1(\tilde{M}/g, \mathbf{Z}_2)$ equals the dimension of $H_c^2(Y, \mathbf{Z}_2)$, while $H^1(\tilde{M}/g, \mathbf{Z}_2)$ had dimension 1. This completes our proof.

Proof of Proposition 1.9. We proceed via proof by contradiction, and assume that $H_c^2(M, \mathbf{Z}_2)$ contains a sub-G-module A whose dimension is finite but greater than 1. Then there exists, by Lemma 1.14, an element g in G such that $\tilde{M}/g$ is an orientable manifold with more than 1 end. Hence there is, by Lemma 1.13, a closed, connected, codimension one submanifold S of $\tilde{M}/g$ which divides $\tilde{M}/g$ into two connected, unbounded components U_1 and U_2 such that the inclusion map of S into $\tilde{M}/g$ induces an isomorphism on fundamental groups.

Let $B = p^{-1}S$, $D_1 = p^{-1}\bar{U}_2$, and $D_2 = p^{-1}\bar{U}_1$, and consider the Mayer–Vietoris sequence for the triad $(\tilde{M}, D_1, D_2)$:

$$\cdots \to H_c^1(B, \mathbf{Z}_2) \to H_c^2(\tilde{M}, \mathbf{Z}_2) \to H_c^2(D_1, \mathbf{Z}_2) \oplus H_c^2(D_2, \mathbf{Z}_2) \to \cdots.$$

Since B is the universal covering space of S, and S is a finite polyhedron with $\pi_1 S$ infinite cyclic, the dimension of $H_c^1(B, \mathbf{Z}_2)$ plus 1 equals the number of ends of $\pi_1 S$, which is 2, and hence $H_c^1(B, \mathbf{Z}_2)$ has dimension 1. Therefore, the above Mayer–Vietoris sequence will give our desired contradiction, provided we can show that A is in the kernel of the map from $H_c^2(\tilde{M}, \mathbf{Z}_2)$ to $H_c^2(D_i, \mathbf{Z}_2)$ for $i = 1$ and 2.

Each finite dimensional subspace of $H_c^2(\tilde{M}, \mathbf{Z}_2)$ is carried by a compact set. Therefore, there exists an open subset V of $\tilde{M}$ such that $\bar{V}$ is compact, and A pulls back to $H_c^2(\tilde{M}, \tilde{M} - V, \mathbf{Z}_2)$. Since G leaves A invariant, A also pulls back to $H_c^2(\tilde{M}, \tilde{M} - hV, \mathbf{Z}_2)$ for each element h of G.

By Lemma 1.11 there exist elements h_1 and h_2 in G such that $h_i\bar{V}$ is a subset of $p^{-1}U_i$ for $i = 1$ and 2, and thus A pulls back to $H_c^2(\tilde{M}, \tilde{M} - p^{-1}U_i, \mathbf{Z}_2)$. Since $\tilde{M} - p^{-1}U_i$ is D_i for $i = 1$ and 2, we see, by considering the long exact sequence in cohomology for the pair $(\tilde{M}, D_i)$, that A maps to zero in $H_c^2(D_i, \mathbf{Z}_2)$.

§2. WHEN THE DIMENSION OF $H^2(G, \mathbf{Z}_2 G)$ IS 1

This section is concerned with the algebraic structure of groups G such that $H^2(G, \mathbf{Z}_2, G)$ has dimension 1.

THEOREM 2.1. *If G is a finitely presented, torsion-free group such that $H^2(G, \mathbf{Z}_2 G)$ has dimension* 1, *then $H^i(G, \mathbf{Z}_2 G)$ vanishes for $i = 0$ and* 1.

Proof. Since G is an infinite group, $H^0(G, \mathbf{Z}_2 G)$ vanishes. Next we assume that $H^1(G, \mathbf{Z}_2 G)$ does not vanish, and show that this leads to a contradiction. By the main theorem of [16], either G is the infinite cyclic group J, or G is a non-trivial free product. But $H^2(J, \mathbf{Z}_2 J)$ vanishes; hence G must be a free product $G_1 * G_2$ where both G_1 and G_2 are infinite groups.

By a result of Stallings, Lemma 1.3 of [20], any retract of a finitely presented group is finitely presented; therefore, the functor $M \to H^2(G_i, M)$ commutes with direct sums. Since $\mathbf{Z}_2 G$ is the direct sum of an infinite number of copies of $\mathbf{Z}_2 G_i$, $H^2(G_i, \mathbf{Z}_2 G)$ has dimension 0 or ∞. But by the "Mayer–Vietoris" sequence, [10] or Theorem 2.3 of [18], $H^2(G, \mathbf{Z}_2 G)$ is isomorphic to $H^2(G_1, \mathbf{Z}_2 G) \oplus H^2(G_2, \mathbf{Z}_2 G)$. This contradicts the fact that $H^2(G, \mathbf{Z}_2 G)$ has dimension 1.

The next theorem is the main result of this section.

THEOREM 2.2. *If G is a finitely presented, torsion-free group such that $H^2(G, \mathbf{Z}_2 G)$ has dimension* 1, *then any finitely generated subgroup T with infinite index in G is free.*

In the following corollary, $\mathbf{Z}$ has the trivial G-module structure.

COROLLARY 2.3. *If G is a finitely presented, torsion-free group such that $H^2(G, \mathbf{Z}_2 G)$ has dimension* 1 *and $H^1(G, \mathbf{Z})$ does not vanish, then the cohomological dimension of G is either* 2 *or* 3.

Proof. The cohomological dimension of G is clearly at least 2. Since $H^1(G, \mathbf{Z})$ does not vanish, G has a normal subgroup H such that G/H is infinite cyclic. Theorem 2.2 yields that H is locally free; hence, by Corollary 2 on page 135 of [7], the cohomological dimension of H is ≤ 2. Therefore, G is an extension of a group of cohomological dimension ≤ 2 by one of cohomological dimension ≤ 1. Thus, by Proposition 9 on page 145 of [7], the cohomological dimension of G is ≤ 3.

Theorem 2.2 is a consequence of the next proposition.

PROPOSITION 2.4. *If G is a finitely presented group such that $H^1(G, \mathbf{Z}_2 G)$ vanishes, and $H^2(G, \mathbf{Z}_2 G)$ has dimension* 1, *then the collection of finitely generated subgroups S of G such that $H^i(S, \mathbf{Z}_2 S)$ vanishes for $i = 0$ and* 1 *is identical with the collection of subgroups which have finite index in G.*

Proof of Theorem 2.2. By [16] and [1] either T is free, or $T = F * G_1 * \ldots * G_n$ where F is free, and each G_j is a finitely generated group such that $H^i(G_j, \mathbf{Z}_2 G_j)$ vanishes for $i = 0$ and 1. Theorem 2.1 yields that $H^1(G, \mathbf{Z}_2 G)$ vanishes; hence, if T is not free, Proposition 2.4 says that each G_i has finite index in G. Since this is impossible, T must be free.

We now state a second consequence of Proposition 2.4.

THEOREM 2.5. *If G is a finitely presented group such that $H^1(G, \mathbf{Z}_2 G)$ vanishes while $H^2(G, \mathbf{Z}_2 G)$ has dimension* 1, *and S is a finitely generated subgroup of G such that S has cohomological dimension* 2, *then S has finite index in G.*

Proof. By [16] and [1] either S is a free group, or $S = F^*G_1^* \dots {}^*G_n$ where F is free, and each G_j is a finitely generated group such that $H^i(G_j, \mathbf{Z}_2 G_j)$ vanishes for $i = 0$ and 1. But, S cannot be free because free groups have cohomological dimension 1. Therefore, Proposition 2.4 yields that G_1 has finite index in G. Thus S has finite index in G.

COROLLARY 2.6. *If G is a finitely presented, torsion-free group such that the dimension of $H^2(G, \mathbf{Z}_2 G)$ is 1, and G contains a finitely generated subgroup of cohomological dimension 2, then G has cohomological dimension 2.*

Proof. Theorem 2.1 says that $H^1(G, \mathbf{Z}_2 G)$ vanishes; hence, Theorem 2.5 yields that G has a subgroup S of finite index such that the cohomological dimension of S is 2. Theorem 1 of [14] states that the cohomological dimension of a torsion free group equals the cohomological dimension of any subgroup of finite index; therefore G has cohomological dimension 2.

Examples 2.7. The only examples known to the author of finitely presented, torsion-free groups G such that $H^2(G, \mathbf{Z}_2 G)$ has dimension 1 are the fundamental groups of closed, two dimensional manifolds with Euler characteristic less than 1.

In order to prove Proposition 2.4, we must first develop some material about the cohomology of a space relative to a family of supports.

Definition 2.8. If $p: E \to B$ is a fiber bundle with fiber F where F, E, and B are countable, locally finite, simplicial complexes, and B is connected, then we define a particular paracompact family of supports θ on E as follows: a set S is an element of θ if and only if p/S is a proper map. We call θ the family of supports determined by p. More generally, if α is a given family of supports on B, then we define the family of supports determined by p and α to be the collection of all elements S in θ such that $p(S)$ is an element of α.

With respect to this definition, we have the following lemma.

LEMMA 2.9. *If α is a paracompact family of supports on B, and Φ denotes the family of supports determined by p and α, then there exists a spectral sequence converging to $H_\Phi{}^*(E, \mathbf{Z}_2)$ such that $E_2{}^{rs}$ is $H_\alpha{}^r(B; H_c{}^s(F, \mathbf{Z}_2))$.*

Remark 2.10. If γ are supports on X, then $H_\gamma{}^*(X, \ )$ denotes cohomology with supports γ and constant coefficients, while $H_\gamma{}^*(X; \ )$ denotes cohomology with supports in γ and local coefficients.

Proof of Lemma 2.9. This lemma is a special case of the Leray spectral sequence. See pages 128 and 129 of [19] for more details.

COROLLARY 2.11. *Let X be a connected, countable, locally finite simplicial complex with fundamental group G, and let θ denote the family of supports determined by the covering projection $p: \tilde{X} \to X$. If $H^i(\tilde{X}, \mathbf{Z})$ vanishes for $0 < i \leq n$, then $H_\theta{}^i(\tilde{X}, \mathbf{Z}_2)$ is isomorphic to $H^i(G, \mathbf{Z}_2 G)$ for all $i \leq n$.*

Proof. Since the fiber F of $p: \tilde{X} \to X$ is the discrete topological space G, $H_c{}^q(F, \mathbf{Z}_2)$ vanishes for $q > 0$, and $H_c{}^o(F, \mathbf{Z}_2)$ in G-isomorphic to $\mathbf{Z}_2 G$. Therefore the spectral sequence of Lemma 2.9 collapses; thus $H_\theta{}^i(\tilde{X}, \mathbf{Z}_2)$ is isomorphic to $H^i(X; \mathbf{Z}_2 G)$. But, by Application 1 on page 356 of [2], $H^i(X; \mathbf{Z}_2 G)$ is isomorphic to $H^i(G, \mathbf{Z}_2 G)$ for all $i \leq n$.

Definition 2.12. Let $p: E \to B$ be a fiber bundle with fiber F and discrete structure group G, such that F, E, and B are countable, locally finite, simplicial complexes with B connected; let α be a family of supports on F which are invariant under the action of G, i.e. if $g \in G$ and $S \in \alpha$ then $gS \in \alpha$. Then, we define supports Ω on E as follows: a set S is an element of Ω if and only if S is a closed subset of a finite union of sets having the form $f(T \times \sigma)$ where T is an element of α, σ is a closed simplex in B, and f is a trivilization of the bundle restricted to σ. We call Ω the family of supports generated by α and p. Note that Ω is a paracompact family whenever α is a paracompact family.

With respect to this definition, we have the next lemma.

LEMMA 2.13. *Let α be a paracompact family of supports on F, and let Ω denote the family of supports generated by α and p, then there exists a spectral sequence converging to $H_\Omega{}^*(E, \mathbf{Z}_2)$ such that $E_2{}^{rs}$ is $H_c{}^r(B; H_\alpha{}^s(F, \mathbf{Z}_2))$.*

Proof. Same as the proof of Lemma 2.9.

Hypothesis 2.14. Let X be a closed, connected, smooth manifold with fundamental group G such that $H^1(G, \mathbf{Z}_2 G)$ vanishes and $H^2(G, \mathbf{Z}_2 G)$ has dimension 1. Also let S be a finitely generated subgroup of G, $\tilde{X}/S$ denote the covering space of X corresponding to S, and $p: \tilde{X} \to \tilde{X}/S$ denote the covering projection.

With respect to this hypothesis, we have the following technical lemma.

LEMMA 2.15. *There exists an open subset U of $\tilde{X}/S$ and an element x in $H_c{}^2(p^{-1}U, \mathbf{Z}_2)$ such that $\bar{U}$ is compact, $H_c{}^i(p^{-1}U, \mathbf{Z}_2)$ vanishes for $i = 0$ and 1, $xs = x$ for each element s of S, and x maps to a non-zero element in $H_c{}^2(\tilde{X}, \mathbf{Z}_2)$ under the map determined by the fact that $p^{-1}U$ is an open subset of $\tilde{X}$.*

Proof. By Proposition 1.2 there exists a non-zero element y in $H_c{}^2(\tilde{X}, \mathbf{Z}_2)$ such that $ys = y$ for each element s in S. Since y is supported by a compact set, there is an open subset V of $\tilde{X}/S$, and an element v in $H_c{}^2(p^{-1}V, \mathbf{Z}_2)$ such that $\bar{V}$ is compact and v maps to y.

Let M denote the sub-S-module of $H_c{}^2(p^{-1}V, \mathbf{Z}_2)$ generated by v, and K equal the intersection of M with the kernel of the map from $H_c{}^2(p^{-1}V, \mathbf{Z}_2)$ to $H_c{}^2(\tilde{X}, \mathbf{Z}_2)$. Then we have the following commutative diagram of S-modules:

$$\begin{tikzcd} K \arrow[r] & M \arrow[r, "\tau"] & \mathbf{Z}_2 \\ \mathscr{K} \arrow[u] \arrow[r] & \mathbf{Z}_2 S. \arrow[u, "\gamma"] \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{2.16}$$

In this diagram τ is the restriction to M of the map from $H_c{}^2(p^{-1}V, \mathbf{Z}_2)$ to $H_c{}^2(\tilde{X}, \mathbf{Z}_2)$, α is the augmentation map whose kernel is denoted by $\mathscr{K}$, and γ is the epimorphism defined by the equation $\gamma(r) = vr$ for all $r \in \mathbf{Z}_2 S$.

Since S is finitely generated, $\mathscr{K}$ is a finitely generated S-module; consequently K is also a finitely generated S-module. Hence there exists an open set W, containing V, such that $\bar{W}$ is compact and K maps to zero under the map from $H_c{}^2(p^{-1}V, \mathbf{Z}_2)$ to $H_c{}^2(p^{-1}W, \mathbf{Z}_2)$. Denote the image of v in $H_c{}^2(p^{-1}W, \mathbf{Z}_2)$ by w, then $ws = w$ for all $s \in S$, and w maps to y.

Let A be the subcomplex of $\tilde{X}/S$, under some triangulation of $\tilde{X}/S$, which consists of all closed simplexes that meet $\overline{W}$; and let B be the subcomplex consisting of all closed simplexes that are disjoint from $\overline{W}$. Then A is a finite complex, $\tilde{X}/S$ is the union of A and B, and $\overline{W}$ is disjoint from B.

By the Mayer–Vietoris sequence, we see that B has only a finite number of connected components. Let D be the union of the unbounded components of B, and let U be the complement of D. Then $\overline{U}$ is compact and W is a subset of U. Denote the image of w in $H_c^2(p^{-1}U, \mathbf{Z}_2)$ by x, then x maps to y, and $xs = x$ for each element s in S. Also, the cohomology exact sequence for the pair $(\tilde{X}, p^{-1}D)$ yields that $H_c^i(p^{-1}U, \mathbf{Z}_2)$ vanishes for $i = 0$ and 1.

Still assuming Hypothesis 2.14, we let Y be the cartesian product of $\tilde{X}$ with itself balanced over S; explicitly Y is the quotient of $\tilde{X} \times \tilde{X}$ via the identifications $(sx, y) = (x, s^{-1}y)$ where $x, y \in X$ and $s \in S$. Then Y is the total space of two fiber bundles each of which has fiber $\tilde{X}$ and base space $\tilde{X}/S$. The first fiber bundle is determined by the projection of $\tilde{X} \times \tilde{X}$ onto its first factor, and the second by the projection of $\tilde{X} \times \tilde{X}$ onto its second factor. We denote the projection map of the first fiber bundle by p_1, and that of the second by p_2.

Let θ be the paracompact family of supports on $\tilde{X}$ determined by the covering projection $p : \tilde{X} \to \tilde{X}/S$. Since θ is invariant under the action of S, we can put on Y the paracompact family of supports Ω generated by θ and p_1.

The proof of Proposition 2.4 is based on the next lemma.

LEMMA 2.17. *The vector space $H_\Omega^2(Y, \mathbf{Z}_2)$ contains a non-zero element.*

Proof of Proposition 2.4. We start by assuming that S is a finitely generated subgroup of G such that $H^i(S, \mathbf{Z}_2 S)$ vanishes for $i = 0$ and 1. Since G is finitely presented, there exists a manifold X satisfying Hypothesis 2.14 relative to the given pair of groups S and G. Then Corollary 2.11, in which X is replaced by $\tilde{X}/S$ and G by S, yields that $H_\theta^i(\tilde{X}, \mathbf{Z}_2)$ vanishes for $i = 0$ and 1.

By Lemma 2.13 there exists a spectral sequence converging to $H_\Omega^*(Y, \mathbf{Z}_2)$ such that E_2^{rs} is $H_c^r(\tilde{X}/S;\ H_\theta^s(\tilde{X}, \mathbf{Z}_2))$; hence $H_\Omega^2(Y, \mathbf{Z}_2)$ is isomorphic to $H_c^0(\tilde{X}/S;\ H_\theta^2(\tilde{X}, \mathbf{Z}_2))$. Therefore, by Lemma 2.17, $H_c^0(\tilde{X}/S;\ H_\theta^2(\tilde{X}, \mathbf{Z}_2))$ contains a non-zero element, but this is impossible unless $\tilde{X}/S$ is compact. Thus S has finite index in G.

On the other hand if we assume that S has finite index in G, then it is easy to show that S is finitely presented and that $H^i(S, \mathbf{Z}_2 S)$ vanishes for $i = 0$ and 1.

Proof of Lemma 2.17. Let U be the open subset of $\tilde{X}/S$ constructed by Lemma 2.15, and let W be the cartesian product of $p^{-1}U$ and $\tilde{X}$ balanced over S. Explicitly, W is the quotient of $p^{-1}U \times \tilde{X}$ via the identifications $(su, y) = (u, s^{-1}y)$ where $u \in p^{-1}U$, $y \in \tilde{X}$ and $s \in S$. Then W is an open subset of Y, $p_2 : W \to \tilde{X}/S$ is a fiber bundle with fiber $p^{-1}U$, and this bundle is a sub-bundle of $p_2 : Y \to \tilde{X}/S$.

If α is a family of supports on a topological space K and A is a subspace of K, then we recall that α/A is the family of supports on A which consists of all elements of α that are subsets of A.

Let Φ be the family of supports on Y determined by $p_2 : Y \to \tilde{X}/S$, then Ω is a subset of Φ. Also using the fact that $\bar{U}$ is compact, it can be shown that Ω/W is equal to Φ/W. In fact Ω/W and Φ/W are both indentical with the family of supports on W determined by the bundle projection $p_2 : W \to \tilde{X}/S$.

Let $g: H^2_{\Omega/W}(W, \mathbf{Z}_2) \to H_\Omega{}^2(Y, \mathbf{Z}_2)$ be the homomorphism which appears in the cohomology exact sequence for the pair of topological spaces $(Y, Y - W)$ relative to the family of supports Ω, and let $h: H_\Omega{}^2(Y, \mathbf{Z}_2) \to H_\Phi{}^2(Y, \mathbf{Z}_2)$ be the homomorphism determined by the fact that $\Omega \subseteq \Phi$. If we can show that the composite of g and h is non-zero, then there must be a non-zero element in $H_\Omega{}^2(Y, \mathbf{Z}_2)$.

Let f_n denote the homomorphism from $H^n_{\Phi/W}(W, \mathbf{Z}_2)$ to $H_\Phi{}^n(Y, \mathbf{Z}_2)$ which appears in the cohomology exact sequence for the pair of topological spaces $(Y, Y - W)$ relative to the family of support Φ. Since Ω/W equals Φ/W, f_2 is the composite of g and h.

We now give an argument to show that f_2 is non-zero; thus proving Lemma 2.17. We start by applying Lemma 2.9 first to the fiber bundle $p_2 : W \to \tilde{X}/S$, and secondly to the bundle $p_2 : Y \to \tilde{X}/S$. In each case we let α be the family of all closed subsets of $\tilde{X}/S$. Thus we obtain two spectral sequences. The first spectral sequence converges to $H^*_{\Phi/W}(W, \mathbf{Z}_2)$, its terms are denoted by $E_n{}^{rs}$, and $E_2{}^{rs}$ is $H^r(\tilde{X}/S; H_c{}^s(p^{-1}U, \mathbf{Z}_2))$. The second spectral sequence converges to $H_\Phi{}^*(Y, \mathbf{Z}_2)$, its terms are denoted by $\mathscr{E}_n{}^{rs}$, and $\mathscr{E}_2{}^{rs}$ is $H^r(\tilde{X}/S; H_c{}^s(\tilde{X}, \mathbf{Z}_2))$.

Since $p_2 : W \to \tilde{X}/S$ is an open sub-bundle of $p_2 : Y \to \tilde{X}/S$, there is a homomorphism of the first spectral sequence to the second which converges to the homomorphism $f_* : H^*_{\Phi/W}(W, \mathbf{Z}_2) \to H_\Phi{}^*(Y, \mathbf{Z}_2)$.

But Lemma 2.15 together with the first spectral sequence yield that $H^2_{\Phi/W}(W, \mathbf{Z}_2)$ can be identified with the subspace A of $H_c{}^2(p^{-1}U, \mathbf{Z}_2)$ consisting of all elements which are left fixed under the action of S. Likewise, Hypothesis 2.14 together with Corollary 2.11 and the second spectral sequence yield that $H_\Phi{}^2(Y, \mathbf{Z}_2)$ can be identified with the subspace of $H_c{}^2(\tilde{X}, \mathbf{Z}_2)$ consisting of all elements left fixed by S.

Let $l: H_c{}^2(p^{-1}U, \mathbf{Z}_2) \to H_c{}^2(\tilde{X}, \mathbf{Z}_2)$ be the homomorphism which appears in the cohomology exact sequence for the pair of topological spaces $(\tilde{X}, \tilde{X} - p^{-1}U)$. Then, under the above identifications, f_2 becomes the restriction of l to A. Let x be the element of $H_c{}^2(p^{-1}U, \mathbf{Z}_2)$ constructed in Lemma 2.15, then $x \in A$ and $l(x) \neq 0$. Thus f_2 is a non-zero homomorphism.

§3. A RESULT ABOUT CRYSTALLOGRAPHIC GROUPS

We now apply the techniques developed in §2 to obtain a result about crystallographic groups. The reader is referred to [22] for the basic facts about these groups.

THEOREM 3.1. *If G is a countable torsion-free group which contains an abelian subgroup S of rank n, and $H^i(G, \mathbf{Z}_2 G)$ vanishes for $i < n$ while $H^n(G, \mathbf{Z}_2 G)$ has dimension 1, then G is a crystallographic group of rank n.*

The following corollary is a consequence of Theorems 2.1 and 3.1 together with the classification of crystallographic groups of rank 2.

COROLLARY 3.2. *If G is a finitely presented torsion-free group which contains an abelian subgroup of rank 2, and $H^2(G, \mathbf{Z}_2 G)$ has dimension 1; then either G is free abelian of rank 2, or G is isomorphic to the non-trivial semi-direct product of the infinite cyclic group with itself.*

Proof of Theorem 3.1. By known results about crystallographic groups, it suffices to show that S has finite index in G.

Since G is a countable group, one can construct a $K(G, 1)$ which is a countable CW-complex. But by Theorem 1 of [13] any countable CW-complex has the homotopy type of a countable, locally finite, simplical complex; hence we can find a countable, locally finite, simplical complex X which is a $K(G, 1)$.

Let T^n denote the cartesian product of n copies of the circle, then S can be identified with the fundamental group of T^n. Since $\mathbf{R}^n$ is the universal cover of T^n, we can form the cartesian product of $\mathbf{R}^n$ with $\tilde{X}$ balanced over S. Denote this space by Y.

Let p denote the covering projection of $\tilde{X}$ onto X, q the projection of $\tilde{X}$ onto $\tilde{X}/S$, and r the projection of $\tilde{X}/S$ onto X. Then use α to denote the paracompact family of supports on $\tilde{X}$ determined by p, while γ denotes the paracompact family of supports on $\tilde{X}/S$ determined by r. Notice that α is the same as the family of supports determined by q and γ.

Now Y fibers over T^n with fiber $\tilde{X}$. Let p_1 denote the projection map in this fibration, and Ω the paracompact family of supports on Y generated by α and p_1. Also Y fibers over $\tilde{X}/S$ with fiber $\mathbf{R}^n$. Let p_2 denote the projection map in this second fibration, and Φ the family of supports on Y determined by p_2 and γ. Then using the fact that T^n is compact, it can be shown that Ω and Φ are the same family of supports.

By Corollary 2.11 $H_\alpha{}^i(\tilde{X}, \mathbf{Z}_2)$ vanishes for $i < n$ and $H_\alpha{}^n(\tilde{X}, \mathbf{Z}_2)$ is $\mathbf{Z}_2$; hence by applying Lemma 2.13 to the fiber bundle $p_1 : Y \to T^n$, we obtain that $H_\Omega{}^n(Y, \mathbf{Z}_2)$ is Z_2.

On the other hand, by applying Lemma 2.9 to the fiber bundle $p_2 : Y \to \tilde{X}/S$ which has supports γ on $\tilde{X}/S$ and Φ on Y, we obtain that $H_\gamma{}^0(\tilde{X}/S; \mathbf{Z}_2)$ is isomorphic to $H_\Phi{}^n(Y, \mathbf{Z}_2)$ which equals $H_\Omega{}^n(Y, \mathbf{Z}_2)$. Thus there exists a non-zero element in $H_\gamma{}^0(\tilde{X}/S, \mathbf{Z}_2)$.

But this is impossible unless $\tilde{X}/S$ is a member of γ, in which case $r: \tilde{X}/S \to X$ is a proper map. In particular, the fiber of r must be compact; hence $\tilde{X}/S$ is a finite sheeted covering space of X. Therefore S has finite index in G.

§4. WHEN THE DIMENSION OF $H^2(G, \mathbf{Z}_2 G)$ IS ∞

In the terminology of [17], $A *_C B$ denotes the free product of A and B with amalgamated subgroup C. Define $[A; B, f]$, when B is a subgroup of A and f an embedding $B \to A$, to be the group obtained from A by adjoining a new generator x and relations $f(b) = xbx^{-1}$ for all b in B. We call $A *_C B$ a non-trivial decomposition if C is a proper subgroup of both A and B. On the other hand, all $[A; B, f]$ are called non-trivial.

Then Stallings proved that any finitely generated group G such that the dimension of $H^1(G, \mathbf{Z}_2 G)$ is ∞ can be non-trivially written as either $A*_F B$ or $[A; F, f]$ where F is a finite group.

This led us to ask the following question. Does every group G such that the dimension of $H^2(G, \mathbf{Z}_2 G)$ is ∞ have a non-trivial decomposition as either $A*_S B$ or $[A; S, f]$ where S comes from some class of infinite groups?

The next theorem gives positive, albeit weak, evidence that the answer to this question is affirmative.

THEOREM 4.1. *If G is a finitely presented, torsion-free group which contains an infinite cyclic, normal subgroup S, and the dimension of $H^2(G, \mathbf{Z}_2 G)$ is ∞; then G can be non-trivially written as either $A*_J B$ or $[A; J, \phi]$ where J denotes an infinite cyclic group containing S.*

Proof. The Lyndon–Hochschild–Serre spectral sequence for the group extension $1 \to S \to G \to G/S \to 1$ converges to $H^*(G, \mathbf{Z}_2 G)$ with E_2^{pq} equal to $H^p(G/S, H^q(S, \mathbf{Z}_2 G))$. Since $H^q(S, \mathbf{Z}_2 G)$ vanishes for $q \neq 1$, this spectral sequence collapses; hence $H^n(G, \mathbf{Z}_2 G)$ is isomorphic to $H^{n-1}(G/S, H^1(S, \mathbf{Z}_2 G))$. But $H^1(S, \mathbf{Z}_2 S)$ is $\mathbf{Z}_2$; therefore $H^1(S, \mathbf{Z}_2 G)$ is isomorphic to $\mathbf{Z}_2 G/S$ as G/S-modules. (To see this, use the formula given in exercise 5 on page 351 of [11]. Also compare the proof of Theorem 3 of [9].) Thus $H^n(G, \mathbf{Z}_2 G)$ is isomorphic to $H^{n-1}(G/S, \mathbf{Z}_2 G/S)$.

In particular $H^2(G, \mathbf{Z}_2 G)$ is isomorphic to $H^1(G/S, \mathbf{Z}_2 G/S)$; hence the dimension of $H^1(G/S, \mathbf{Z}_2 G/S)$ is ∞. Thus Stallings' result allows us to write G/S non-trivially as either $X*_F Y$ or $[X; F, f]$ where F is a finite group.

Let p denote the quotient map from G onto G/S and let J denote $p^{-1}F$. Then J is infinite cyclic by 5.2 of [16].

When G/S is written as $X*_F Y$, let A be $p^{-1}X$ and B be $p^{-1}Y$, then B can be written non-trivially as $A*_J B$. (To see this, use Corollary 4.4.1 of [12].)

On the other hand when G/S is written as $[X; F, f]$, let A be $p^{-1}X$ and let x be an element in G such that $p(x)$ is the generator adjoined to X to form $[X; F, f]$. Define $\phi(n)$ to equal xnx^{-1} for all $n \in J$. Then G can be written as $[A; J, \phi]$. (To see this, use the results on pp. 41 and 42 of [3].)

§5. APPENDIX

In this section we sketch proofs of the following extensions of Theorem 1.5.

THEOREM 5.1. *If G is a finitely presented group which contains at least one element of infinite order, and K is a field with non-zero characteristic, then any sub-G-module of $H^2(G, KG)$ has K-dimension* 0, 1, *or* ∞.

COROLLARY 5.2. *If G is a finitely presented group which contains at least one element of infinite order, then the abelian group $H^2(G, \mathbf{Z}G)$ either vanishes, is infinite cyclic, or is not finitely generated.*

In the next theorem $\mathbf{Q}$ denotes the field of rational numbers.

THEOREM 5.3. *If G is a finitely presented, torsion-free group whose cohomological dimension is ∞, then any sub-G-module of $H^2(G, \mathbf{Q}G)$ has $\mathbf{Q}$-dimension* 0, 1, *or* ∞.

Let K be a field of non-zero characteristic p, then the proof of Theorem 5.1 is based on the following elementary lemma. (All tensor products used in this lemma are over the prime field $\mathbf{Z}_p$.)

LEMMA 5.4. *If V is a $\mathbf{Z}_p[x, x^{-1}]$ module such that $V \otimes K$ contains a sub-$K[x, x^{-1}]$-module A whose K-dimension is finite, then there exists a positive integer m such that x^m acts trivially on A.*

Proof. Clearly V contains a finitely generated sub-$\mathbf{Z}_p[x]$-module W such that $A \subseteq W \otimes K$. Since $\mathbf{Z}_p[x]$ is a principal ideal domain, W is a direct sum of cyclic submodules. If C is a free summand of W, then A must project into the zero submodule of $C \otimes K$; hence we may assume that W has finite $\mathbf{Z}_p$-dimension s. Therefore the action of x on W determines an element of the finite group $GL_s(\mathbf{Z}_p)$; thus there exists a positive integer m such that x^m acts trivially on A.

Proof of Theorems 5.1 *and* 5.3. We argue, in both cases, as we did in the proof of Theorem 1.5 pointing out where modifications are necessary.

Let M be a closed, connected, smooth manifold whose dimension is bigger than 5 and whose fundamental group is G. Since Proposition 1.2 remains valid when Z_2 is replaced by any field F, $H^2(G, FG)$ is isomorphic to a sub-G-module of $H_c{}^2(\tilde{M}, F)$.

Therefore we need only prove Proposition 1.9 with $\mathbf{Z}_2$ replaced by F. Here everything goes through as before except for the proof of Lemma 1.14 where there are two difficulties.

One difficulty is to conclude that $\tilde{M}/g$ has more than 1 end from the fact that the F-dimension of $H_c{}^1(\tilde{M}/g, F)$ is greater than 1. But this difficulty can be resolved for any field F by using Theorem 1.9 of [15].

The other, apparently, serious difficulty is to find a positive integer m such that h^m acts trivially on A.

To prove Theorem 5.1, we resolve this difficulty by using Lemma 5.4 with V equal to $H_c{}^2(\tilde{M}, \mathbf{Z}_p)$ and the action of x identified with that of h.

On the other hand to prove Theorem 5.3, we modify the first 2 sentences in the proof of Lemma 1.14 by picking h to be any element different from the identity in the kernel of the action of G on A. Such an element exists by Theorem 5 of [14].

Proof of Corollary 5.2. Assume that $H^2(G, \mathbf{Z}G)$ is a finitely generated, abelian group. (In what follows, $\otimes$ and Tor are over the ring $\mathbf{Z}$.) Then for each prime p, $H^2(G, \mathbf{Z}G) \otimes \mathbf{Z}_p$ is a finite dimensional $\mathbf{Z}_p$ vector space and a sub-G-module of $H^2(G, \mathbf{Z}_p G)$. Hence by Theorem 5.1 the $\mathbf{Z}_p$-dimension of $H^2(G, \mathbf{Z}G) \otimes \mathbf{Z}_p$ is 0 or 1. From this we easily conclude that $H^2(G, \mathbf{Z}G)$ is cyclic. If $H^2(G, \mathbf{Z}G)$ is neither 0 nor infinite cyclic, then there is a prime p such that Tor $(H^2(G, \mathbf{Z}G), \mathbf{Z}_p)$ does not vanish, which would contradict Corollary 3.7 of [18].

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